

Transfer Matrix Functional Relations for the Generalized $\tau_2(t_q)$ Model

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The N -state chiral Potts model in lattice statistical mechanics can be obtained as a “descendant” of the six-vertex model, via an intermediate “ Q ” or “ $\tau_2(t_q)$ ” model. Here we generalize this to obtain a column-inhomogeneous $\tau_2(t_q)$ model, and derive the functional relations satisfied by its row-to-row transfer matrix. We do *not* need the usual chiral Potts relations between the N th powers of the rapidity parameters a_p, b_p, c_p, d_p of each column. This enables us to readily consider the case of fixed-spin boundary conditions on the left and right-most columns. We thereby re-derive the simple direct product structure of the transfer matrix eigenvalues of this model, which is closely related to the superintegrable chiral Potts model with fixed-spin boundary conditions.

KEY WORDS: Statistical mechanics; lattice models; chiral Potts model; six-vertex model; Q matrix.

1. INTRODUCTION

In a remarkable paper,⁽¹⁾ Bazhanov and Stroganov showed in 1990 how the recently-discovered solvable chiral Potts model could be obtained from the six-vertex model by a two-stage process. First one looked for a “ Q ” or $\tau_2(t_q)$ model whose column-to-column transfer matrix commuted with that of the six-vertex model. This turned out to be a spin model on the square lattice, each spin taking a given number N of values. Then one looked for a third model whose row-to-row transfer matrix commuted with that of the $\tau_2(t_q)$ model. This was the N -state chiral Potts model. Some of this working was re-presented and extended by Baxter, Bazhanov and Perk.^(2,3)

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Here we focus attention on the square lattice \mathcal{L} of L columns. With row i we associate a horizontal “rapidity” q_i . With column j we associate two successive vertical rapidities p_{2j-1}, p_{2j} , as in Fig. 1 (except for the initial six-vertex model, which has only one rapidity line per column, as we mention below).

The p and q rapidities are of different types: q is a six-vertex model rapidity, specified by a single complex variable t_q , while p is a chiral Potts model rapidity, specified by four homogeneous variables a_p, b_p, c_p, d_p . Because the aim of the earlier papers was to establish a link between the six-vertex and homogeneous chiral Potts models, the chiral Potts conditions⁽⁴⁾

$$a_p^N + k'b_p^N = kd_p^N, \quad k'a_p^N + b_p^N = kc_p^N, \quad (1)$$

were immediately introduced. Here k, k' are fixed constants (the same for all sites and edges of the lattice), satisfying

$$k^2 + k'^2 = 1. \quad (2)$$

The point of this paper is to emphasize that these conditions are *not* needed at the first step of the procedure. Even without them, and with a_p, b_p, c_p, d_p allowed to take arbitrary values for each of the $2L$ vertical rapidity lines of the lattice, it is still true that the column-to-column transfer matrix (the “ Q ” matrix) of the $\tau_2(t_q)$ model commutes with that of the original six-vertex model (though not with one another). Further, the row-to-row transfer matrices of two models $\tau_2(t_q), \tau_2(t_q')$ (with different horizontal rapidity variables t_q, t_q' but the same vertical p -rapidities) commute. These $\tau_2(t_q)$ and $\tau_j(t_q)$ matrices satisfy straightforward generalizations of the functional relations (4.27) of ref. 2.

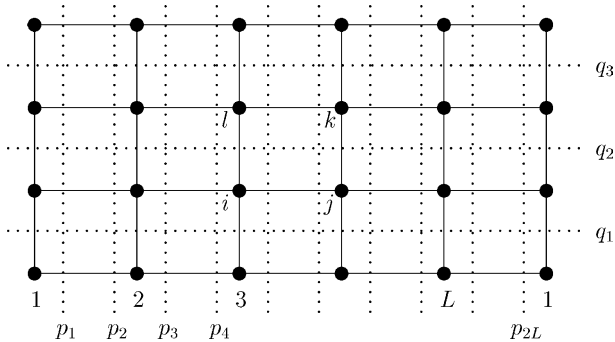


Fig. 1. The square lattice \mathcal{L} of L columns with cyclic boundary conditions, showing the horizontal rapidity lines q_1, q_2, \dots and the vertical rapidity lines p_1, p_2, \dots, p_{2L} .

We shall also show, still *without* the conditions (1), that we can define a chiral Potts model that is related to the $\tau_2(t_q)$ model by appropriate generalizations of the transfer matrix functional relations of ref. 2. It is, however, in general inhomogeneous, its Boltzmann weights being of the usual form, but the parameters therein being related in a rather complicated algebraic manner both to the p variables and to the t_q variable. Further, its row-to-row transfer matrices do not in general commute with one another, except in the particular combination

$$T(\omega^i x, \omega^j y) \widehat{T}(\omega^k x, \omega^l y)$$

defined in Eqs. (25)–(28), (39)–(41) below.² (Here i, j, k, l are arbitrary integers.) Such a two-row transfer matrix is basically that of the “superintegrable” chiral Potts model.⁽⁵⁾

Finally, we shall remark that this general inhomogeneous model includes as a special case the homogeneous $\tau_2(t_q)$ model with closed (fixed-spin) boundary conditions. The functional relations between the $\tau_j(t_q)$ matrices then simplify, and the eigenvalue spectrum is that of a direct product of L single-spin matrices. This agrees with the properties of such a superintegrable chiral Potts model that we observed in ref. 5.

We have written the conditions (1), (2) down because they are so usually associated with the chiral Potts model.³ Here we *never* use them. The rapidity $p = \{a_p, b_p, c_p, d_p\}$ has value $p(m) = \{a_{p(m)}, b_{p(m)}, c_{p(m)}, d_{p(m)}\}$ on vertical rapidity line m , for $m = 1, 2, \dots, 2L$. There is no restriction on the complex numbers $a_{p(m)}, b_{p(m)}, c_{p(m)}, d_{p(m)}$.

2. THE SIX-VERTEX MODEL

We start by defining a six-vertex model in a particular field. For this model the doubled vertical rapidity lines p_1, \dots, p_{2L} in Fig. 1 should be replaced by single “type q ” rapidity lines r_1, \dots, r_L .

Associate a spin σ_i with each site i of the square lattice \mathcal{L} , and allow σ_i to take N successive integer values, say $1, 2, \dots, N$. These can be extended to all integer values with the modular N convention $\sigma_i = \sigma_i + N$. Not all values are allowed: vertically adjacent spins σ_j, σ_k , with k above j as in Fig. 1, must satisfy the adjacency rule:

$$\sigma_k = \sigma_j \text{ or } \sigma_j - 1, \quad \text{mod } N. \quad (3)$$

²The $\widehat{T}(x, y)$ of this paper generalizes the \widehat{T} of ref. 2, but with x and y interchanged.

³They ensure that the column-to-column transfer matrices of the $\tau_2(t_q)$ model, i.e. the Q model, commute.

and horizontally adjacent spins σ_i, σ_j , with j to the right of i , must satisfy

$$\sigma_j = \sigma_i \text{ or } \sigma_i + 1, \pmod{N}. \quad (4)$$

A typical face i, j, k, l , with the corner sites i, j, k, l arranged anti-clockwise from the bottom-left of the square lattice is shown in Fig. 1. With each face i, j, k, l associate a Boltzmann weight function $W_{6V}(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$. If $\sigma_i = a$ is fixed, then there are only six possible choices of the other three spins, as shown in Fig. 1. We define the corresponding Boltzmann weights to be

$$\begin{aligned} W_{6V}(a, a, a, a) &= \omega t - 1, & W_{6V}(a, a + 1, a, a - 1) &= \omega t - 1, \\ W_{6V}(a, a + 1, a - 1, a) &= t - 1, & W_{6V}(a, a, a - 1, a - 1) &= \omega(t - 1), \\ W_{6V}(a, a + 1, a, a) &= \omega - 1, & W_{6V}(a, a, a, a - 1) &= (\omega - 1)t, \end{aligned} \quad (5)$$

for all integers a . Here

$$\omega = e^{2\pi i/N} \quad (6)$$

and t is a free parameter. For all other (non-allowed) values of a, b, c, d we take $W_{6V}(a, b, c, d)$ to be zero. We may exhibit the t dependence by writing $W_{6V}(a, b, c, d)$ as $W_{6V}(t|a, b, c, d)$.

The partition function is

$$Z = \sum \prod_{ijkl} W_{6V}(\sigma_i, \sigma_j, \sigma_k, \sigma_l). \quad (7)$$

Here the product is over all faces (i, j, k, l) of the lattice, with cyclic (toroidal) boundary conditions. The outer sum is over all values of all the spins.

One can regain the usual arrow picture of the six-vertex model by drawing arrows on the edges of the dual lattice, pointing to the left or up if the spins on either sides are equal, to the right or down else. Then the six spin configurations of Fig. 2 become those of Fig. 8.2 of ref. 6, where two arrows point into each vertex and two point out. Note that the

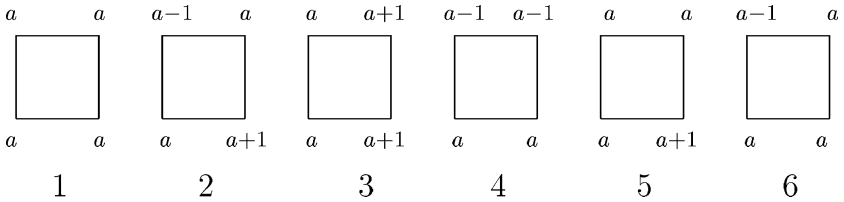


Fig. 2. The six spin configurations of the six-vertex model.

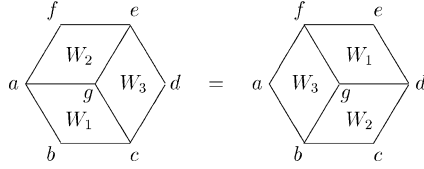


Fig. 3. The generalized star-triangle relation.

weights (5) are not those of the usual zero-field six-vertex model, since the weights of the third and fourth configurations are unequal. A field with weight $\omega^{1/2}$ has been applied to the third and fourth weights.⁴ Apart from this field, the λ, v of Eq. (9.2.3) of ref. 6 are related to our present variables ω by $e^{-2\lambda} = \omega$, $e^{\lambda+v} = t$.

With each horizontal (vertical) rapidity q_i (r_j) we associate a parameter t_{q_i} , t_{r_j} . Then the model is solvable if for each face

$$t = t_q/t_r, \quad (8)$$

$q = q_i$ being the horizontal rapidity and $r = r_j$ the vertical. This is in part because the function W_{6V} satisfies the star-triangle relation

$$\begin{aligned} & \sum_g W_{6V}(t_q|b, c, g, a) W_{6V}(t_r|a, g, e, f) W_{6V}(t_r/t_q|g, c, d, e) \\ &= \sum_g W_{6V}(t_r/t_q|a, b, g, f) W_{6V}(t_r|b, c, d, g) W_{6V}(t_q|g, d, e, f) \end{aligned} \quad (9)$$

for all t_q, t_r and all values of the external spins a, b, c, d, e, f . This relation is depicted graphically in Fig. 3, provided we take W_1, W_2, W_3 therein to be $W_{6V}(t_q), W_{6V}(t_r), W_{6V}(t_r/t_q)$.

2.1. Transfer Matrices

In this N -state spin formulation, the row-to-row transfer matrix of the six-vertex model is an N^L by N^L matrix U_{6V} , with entries

$$[U_{6V}]_{a,b} = \prod_{j=1}^L W_{6V}(a_j, a_{j+1}, b_{j+1}, b_j), \quad (10)$$

⁴Pasquier and Saleur consider the hamiltonian associated with the six-vertex model in this special field.⁽⁷⁾

where $a = \{a_1, \dots, a_L\}$ is the set of spins in the lower of two successive rows of the lattice \mathcal{L} , and $b = \{b_1, \dots, b_L\}$ is the set of spins in the row immediately above. It depends on t , so can be written as $U_{6V}(t)$.

Similarly, the column-to-column transfer matrix is $V_{6V} = V_{6V}(t)$, where

$$[V_{6V}]_{a,b} = \prod_{j=1}^M W_{6V}(a_j, b_j, b_{j+1}, a_{j+1}), \quad (11)$$

and $a = \{a_1, \dots, a_M\}$ is the set of spins in one column, $b = \{b_1, \dots, b_M\}$, is the set of spins one column to the right, and M is the number of rows of the lattice.

Regarding the spins b, c, e, f as fixed, the star-triangle relation (9) can be viewed as the element (a, d) of an N by N matrix relation. It involves matrices with entries $W_3(a, c, d, e)$, $W_3(a, b, d, f)$. Provided these are invertible (which they usually are), the relation ensures that^(6, Sec. 9.6).

$$U_{6V}(t_q) U_{6V}(t_r) = U_{6V}(t_r) U_{6V}(t_q), \quad (12)$$

i.e. the row-to-row transfer matrices commute, for all choices of t_q, t_r .

Similarly, regarding a, c, d, f as fixed and each side of (9) as the element (b, e) of an N by N matrix, it also implies that

$$V_{6V}(t_q) V_{6V}(t_r) = V_{6V}(t_r) V_{6V}(t_q), \quad (13)$$

so the column-to-column transfer matrices also commute with one another.

3. THE $\tau_2(t_q)$ MODEL

The $\tau_2(t_q)$ model is also an N -state model on the square lattice \mathcal{L} , but now the spins only need to satisfy the vertical adjacency rule (3). The horizontal rule (4) is *not* imposed. The partition function is again given by (7), but with W_{6V} replaced by W_τ .

The vertical adjacency rule means that the Boltzmann weight function $W_\tau(a, b, c, d)$ is zero unless $a - d = 0$ or $1 \pmod{N}$ and $b - c = 0$ or $1 \pmod{N}$. If these constraints are satisfied, then

$$\begin{aligned} W_\tau(a, b, c, d) &= W_\tau(t_q | a, b, c, d) \\ &= \sum_m \omega^{m(d-b)} (-\omega t_q)^{a-d-m} F_{pq}(a-d, m) F_{p'q}(b-c, m), \end{aligned} \quad (14)$$

where

$$\begin{aligned} F_{pq}(0, 0) &= 1, & F_{pq}(0, 1) &= -\omega c_p t_q / b_p, \\ F_{pq}(1, 0) &= d_p / b_p, & F_{pq}(1, 1) &= -\omega a_p / b_p. \end{aligned} \quad (15)$$

This function W_τ is the multiplicand of eqn. (3.44a) of ref. 2. It is linear in the rapidity variable t_q and is the Boltzmann weight of two triangles $\{a, d, m\}$, $\{b, c, m\}$, summed over the common spin m , with value 0 or 1, as represented in Fig. 4. The triangles have weights $F_{pq}(a-d, m)$, $F_{p'q}(b-c, m)$. There are also edge weights ω^{md} , ω^{-mb} , $(-\omega t_q)^{a-d}$, and a site weight $(-\omega t_q)^{-m}$.

Here p denotes the four complex variables a_p, b_p, c_p, d_p . Auxiliary variables that we shall use are

$$x_p = a_p / d_p, \quad y_p = b_p / c_p, \quad t_p = x_p y_p, \quad \mu_p = d_p / c_p. \quad (16)$$

Similarly for p' . Throughout this paper we impose no restrictions on a_p, b_p, c_p, d_p (or $a_{p'}, b_{p'}, c_{p'}, d_{p'}$). They are *independent* variables.

In (14) p and p' are the values of p for the particular face of the lattice under consideration. If the face is between spin columns J and $J+1$, then $p = p_{2J-1}$ and $p' = p_{2J}$.

We see the reason for the doubling of the vertical rapidity lines in Fig. 1. The odd rapidities $p_1, p_3, \dots, p_{2L-1}$ are those of triangles such as the one on the left in Fig. 4, with weight F_{pq} . The even rapidities p_2, p_4, \dots, p_{2L} are those of triangles on the right in Fig. 4, with weight $F_{p'q}$.

The remarkable feature of the $\tau_2(t_q)$ model is that W_τ and W_{6V} together satisfy a second star-triangle relation:

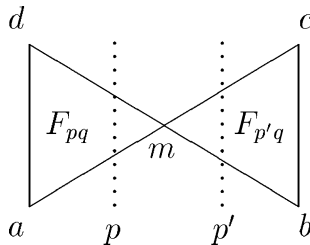


Fig. 4. The Boltzmann weight $W_\tau(a, b, c, d)$ of the τ_2 model as that of two osculating triangles. If a, d are in column J and b, c are in column $J+1$, then $p = p_{2J-1}$ and $p' = p_{2J}$.

$$\begin{aligned} & \sum_g W_\tau(t_q|b, c, g, a)W_\tau(t_r|a, g, e, f)W_{6V}(t_r/t_q|g, c, d, e) \\ &= \sum_g W_{6V}(t_r/t_q|a, b, g, f)W_\tau(t_r|b, c, d, g)W_\tau(t_q|g, d, e, f) \end{aligned} \quad (17)$$

for all t_q, t_r .

This can be obtained from (9) simply by replacing the first two W_{6V} functions on each side by W_τ . It is also represented by Fig. 3, but now W_1, W_2, W_3 therein should be replaced by $W_\tau(t_q), W_\tau(t_r), W_{6V}(t_r/t_q)$.

We can define row-to-row and column-to-column transfer matrices $\tau_2(t_q), V_\tau(p, p')$ for the τ_2 model by replacing W_{6V} in (10) and (11) by W_τ . Then (17) implies that

$$\tau_2(t_q)\tau_2(t_r) = \tau_2(t_r)\tau_2(t_q), \quad (18)$$

i.e. the row-to-row transfer matrices commute for all t_q .

Also, from (17),

$$V_{6V}(t)V_\tau(p, p') = V_\tau(p, p')V_{6V}(t), \quad (19)$$

for all choices of t, p, p', t_q in (5), (14), (15). The column-to-column transfer matrices of the six-vertex model therefore commutes with that of any particular τ_2 model. This was the starting-point of Bazhanov and Stroganov's derivation.⁽¹⁾ Note that it does *not* imply that the column-to-column transfer matrices of two different τ_2 models commute. This is because when ω has the particular "root of unity" value (6) the eigenvalues of the six-vertex model are degenerate.

We shall not consider column-to-column transfer matrices any further herein. All the transfer matrices we shall write down in subsequent equations will be row-to-row matrices of dimension N^L by N^L . The vertical p -rapidities are to be regarded as constants and the horizontal q -rapidity parameters t_q as variables, in general complex.

We note in passing that a useful check on both star-triangle relations is provided by noting from (5) that

$$W_{6V}(1|a, b, c, d) = \delta(a, c).$$

When $t_q = t_r$, it immediately follows that $g = d$ ($g = a$) on the left hand side (LHS) (right hand side (RHS)) of each relation, and that both are trivially satisfied.

Evaluating (14) for the four values of $a-d$ and $b-c$, we find that in each case it is linear in the variable t_q , so from (10) the matrix $\tau_2(t_q)$ is a polynomial in t_q of degree at most L (usually it is of degree L). From the commutation relation (18) (assuming, as seems to be the case, that the eigenvalues are not identically degenerate), the eigenvectors of $\tau_2(t_q)$ must be independent of t_q . The eigenvalues are therefore also polynomials in t_q of degree L .

4. THE τ_2, T RELATION

The commutation relation (18) is true for any two τ_2 models with the same vertical rapidities p_1, p_2, \dots, p_{2L} and different horizontal rapidities t_q, t_r . Here p_j is short-hand for the set $\{a_{p_j}, b_{p_j}, c_{p_j}, d_{p_j}\}$, so there are actually $8L$ complex numbers specifying the vertical rapidities. We emphasize that there are *no* constraints on these numbers. They can *all* be chosen *independently* and (18) will still be satisfied.

The object of this paper is to generalize the transfer matrix functional relations of ref. 2 to this arbitrary inhomogeneous model. We start with the τ_2, T relation of section 4 therein. All equation numbers herein that contain a decimal point, e.g. (4.10), are references to equations of ref. 2.

Without loss of generality, we can take $k=0$ in ref. 2. Then (4.4) becomes

$$[G_J(a)]_{m,m'} = \sum_d \omega^{m'd-ma} (-\omega t_q)^{a-d-m'} F_{pq}(a-d, m') F_{p'q}(a-d, m) g_J(d) \quad (20)$$

and (4.9) is

$$\sum_{m'=0}^1 [G_J(a)]_{m,m'} (-r_{J+1})^{m'} = g'_J(a) (-r_J)^m. \quad (21)$$

Here m, m' take the values 0, 1 and the sum in (20) is over the allowed values $a, a-1$ of the spin d .

The RHS of (20) is the Boltzmann weight of two successive triangles of the τ_2 model, as shown in Fig. 5, with associated edge and site weights, and an additional site weight $g_J(d)$. The spins a, d are in column J ($J=1, \dots, L$) of the lattice shown in Fig. 1, so the p, p' here are the rapidities p_{2J-1}, p_{2J-2} :

$$p = p_{2J-1}, \quad p' = p_{2J-2}. \quad (22)$$

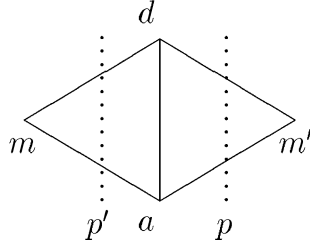


Fig. 5. The sites and faces of \mathcal{L} involved in the working of section 4 of ref. 2. If a, d are in column J , then $p' = p_{2J-2}$ and $p = p_{2J-1}$.

With this identification, (4.11) follows and (4.12) is

$$\frac{a_{p_{2J-2}}^N - d_{p_{2J-2}}^N r_J^N}{c_{p_{2J-2}}^N t_q^N - b_{p_{2J-2}}^N r_J^N} \times \frac{d_{p_{2J-1}}^N t_q^N - a_{p_{2J-1}}^N r_{J+1}^N}{b_{p_{2J-1}}^N - c_{p_{2J-1}}^N r_{J+1}^N} = 1 \quad (23)$$

for $J = 1, \dots, L$, taking $p_0 = p_{2L}$. This is the condition that the function $g_J(a)$ be periodic of period N , i.e. $g_J(a + N) = g_J(a)$. We need this because the spins in the τ_2 model only take N values and are always to be interpreted as integers modulo N .

Given r_1^N , we can solve the bilinear relation (23) successively for $r_2^N, \dots, r_L^N, r_{L+1}^N$. Since $r_{L+1} = r_1$, this gives a quadratic relation for r_1^N , and hence for all the r_J^N . In ref. 2, we then used the fact that we were taking the vertical rapidities to be those of the usual chiral Potts model to obtain the two explicit solutions (4.13) for the r_J^N .

Here we can no longer do this: all we can say is that the sequence r_1^N, \dots, r_L^N is one of the two solutions of (23).⁵

We can still carry on with the rest of the working. In place of (4.13) we set

$$r_J = \omega^{1-\beta_{J-1}}, \quad x_{J-1}, \quad y_{J-1} = t_q/x_{J-1}, \quad (24)$$

where β_{J-1} is an integer that does not enter the relations (23). For given vertical rapidities p_1, \dots, p_L , we take $t_q, x_1, \dots, x_L, y_1, \dots, y_L$ to be fixed, satisfying (23) and (24). Then we allow β_1, \dots, β_L to take any set of integer values.

Equations (4.15)–(4.18) still follow, provided we replace $\omega^{-\beta_J} a_q/d_q$ by $\omega^{-\beta_J} x_J$, $\omega^{-\beta_J} c_q/b_q$ by $\omega^{-\beta_J}/y_J$, and similarly with J replaced by $J-1$. Then in place of (4.19) we obtain

⁵They do have some interesting properties, as in Eq. (55).

$$\begin{aligned}
 g_J(a) &= y_p y_{p'} \overline{W}_{J-1}(a - \beta_{J-1} | \omega x_{J-1}, y_{J-1}) W_J(a - \beta_J | \omega x_J, y_J), \\
 g'_J(a) &= \frac{(y_p - \omega x_J)(t_{p'} - t_q)}{x_{p'} - x_{J-1}} \\
 &\quad \times \overline{W}_{J-1}(a - \beta_{J-1} | x_{J-1}, y_{J-1}) W_J(a - \beta_J | x_J, y_J), \\
 g''_J(a) &= \frac{(1 - y_{p'}/y_{J-1})(t_p - \omega t_q)}{1 - x_p/y_J} \overline{W}_{J-1}(a - \beta_{J-1} | \omega x_{J-1}, \omega y_{J-1}) \\
 &\quad \times W_J(a - \beta_J | \omega x_J, \omega y_J).
 \end{aligned}$$

Here $p = p_{2J-1}$, $p' = p_{2J-2}$ and $x_p, y_p, t_p, x_{p'}, y_{p'}, t_{p'}$ are defined by (16). The functions W, \overline{W} are given by

$$W_J(a | x_J, y_J) = \prod_{i=1}^a \frac{d_{p_{2J-1}} - \omega^i a_{p_{2J-1}}/y_J}{b_{p_{2J-1}} - \omega^i c_{p_{2J-1}} x_J} \quad (25)$$

$$\overline{W}_J(a | x_J, y_J) = \prod_{i=1}^a \frac{\omega a_{p_{2J}} - \omega^i d_{p_{2J}} x_J}{c_{p_{2J}} - \omega^i b_{p_{2J}}/y_J} \quad (26)$$

Note the distinction between x and y with a p or p' suffix, and x and y with an integer J or $J-1$ suffix. The former are defined by (16) and are vertical rapidity variables. The latter are defined by (24). In fact, x_J, y_J are generalizations of the x_q, y_q of ref. 2, so can be thought of as “ q variables”, but they also depend via (23) on all the vertical rapidities.

These functions are generalizations of the chiral Potts model edge-weight functions W, \overline{W} ^{(4,Eqs. (2) and (3);8, Eqs. (2.4) and (2.5))}. Their definitions can be extended to negative integers a in the usual way:

$$\prod_{i=1}^a s_i = \prod_{i=a+1}^0 1/s_i$$

for any s_i . However, they do *not* satisfy the usual periodicity conditions $W(a+N) = W(a)$, $\overline{W}(a+N) = \overline{W}(a)$. Instead they satisfy the weaker condition

$$\frac{W_J(a+N | x_J, y_J) \overline{W}_{J-1}(b+N | x'_{J-1}, y'_{J-1})}{W_J(a | x_J, y_J) \overline{W}_{J-1}(b | x'_{J-1}, y'_{J-1})} = 1 \quad (27)$$

where $x'_j = \omega^j x_J$, $y'_j = \omega^j y_J$ and the integers a, b, i, j are arbitrary. In fact this is just the condition (23). It ensures that the functions $g_J(a)$, $g'_J(a)$, $g''_J(a)$ are all periodic of period N .

4.1. The “Chiral Potts” Transfer Matrix

We now generalize the usual definition of the chiral Potts transfer matrix in (2.15a) and define a matrix $T(x, y)$ with entries

$$T(x, y)_{a, \beta} = \prod_{J=1}^L W_J(a_J - \beta_J | x_J, y_J) \overline{W}_{J-1}(a_J - \beta_{J-1} | x_{J-1}, y_{J-1}). \quad (28)$$

Here $x = \{x_1, \dots, x_L\}$, $y = \{y_1, \dots, y_L\}$, $a = \{a_1, \dots, a_L\}$ and $\beta = \{\beta_1, \dots, \beta_L\}$.

Because of (27), incrementing any of the spins a_1, \dots, a_L by N leaves $T(x, y)_{a, \beta}$ unchanged. Thus the rows of the matrix $T(x, y)$ have the same modulo- N spin invariance property as the rows and columns of the τ_2 -model transfer matrix $\tau_2(t_q)$. Restricting each of these spins to N values, $\tau_2(t_q)$ is a square N^L by N^L matrix; $T(x, y)$ has N^L rows.

The columns of $T(x, y)$, labelled by β_1, \dots, β_L , are slightly more subtle. Incrementing any β_J by N does change the RHS of (28), but only by multiplying it by a factor independent of a_1, \dots, a_L . Further, this factor depends on x_1, \dots, x_L only via their N th powers. It follows that $T(x, y)$ has at most N^L linearly independent columns, and numerical calculations strongly suggest that in general there are indeed N^L linearly independent columns. Thus although we may take $T(x, y)$ to have more than N^L columns, there is a unique N^L by N^L matrix $S_{ij}(x, y)$ such that

$$S_{ij}(x, y) T(x, y) = T(\omega^i x, \omega^j y) \quad (29)$$

for all integers i, j . We formally write $S_{ij}(x, y)$ as

$$T(\omega^i x, \omega^j y) T(x, y)^{-1}.$$

With these definitions, Eq. (4.20) of ref. 2 becomes

$$\tau_2(t_q) T(\omega x, y) = c(x, t_q) T(x, y) + d(y, t_q) T(\omega x, \omega y), \quad (30)$$

where

$$\begin{aligned} c(x, t_q) &= \prod_{J=1}^L \frac{(y_{p_{2J-1}} - \omega x_J)(t_{p_{2J}} - t_q)}{y_{p_{2J-1}} y_{p_{2J}} (x_{p_{2J}} - x_J)}, \\ d(y, t_q) &= \prod_{J=1}^L \frac{(y_{p_{2J}} - y_J)(t_{p_{2J-1}} - \omega t_q)}{y_{p_{2J-1}} y_{p_{2J}} (x_{p_{2J-1}} - y_J)}. \end{aligned} \quad (31)$$

As in (2.42), define an N^L by N^L matrix X with entries

$$X_{\sigma, \sigma'} = \prod_{J=1}^L \delta(\sigma_J, \sigma'_J + 1). \quad (32)$$

This is the operator that shifts all spins in a row by one. It commutes with $\tau_2(t_q)$:

$$X \tau_2(t_q) = \tau_2(t_q) X. \quad (33)$$

Replacing x_J, y_J in (25), (26) by $\omega^{-1}x_J, \omega y_J$ is equivalent to replacing the index i by $i - 1$. It follows that

$$T(\omega^{-1}x, \omega y) = \rho(x, y) X T(x, y), \quad (34)$$

where

$$\rho(x, y) = \prod_{J=1}^L \frac{\mu_{p_{2J-1}} \mu_{p_{2J}} (1 - x_{p_{2J-1}}/y_J)(\omega x_{p_{2J}} - x_J)}{(y_{p_{2J-1}} - x_J)(1 - y_{p_{2J}}/y_J)}. \quad (35)$$

One other function that we need is given by the obvious generalization of (4.23):

$$z(t_q) = \prod_{J=1}^L \omega \mu_{p_{2J-1}} \mu_{p_{2J}} (t_{p_{2J-1}} - t_q)(t_{p_{2J}} - t_q) / (y_{p_{2J-1}} y_{p_{2J}})^2. \quad (36)$$

It is a polynomial in t_q , of degree $2L$. Then one can verify that

$$c(x, \omega t_q) d(y, t_q) \rho(\omega x, y) = z(\omega t_q) \quad (37)$$

from which it follows that we write (30) as

$$\tau_2(t_q) T(\omega x, y) = \frac{z(t_q)}{d(\omega^{-1}y, \omega^{-1}t_q)} X T(\omega x, \omega^{-1}y) + d(y, t_q) T(\omega x, \omega y). \quad (38)$$

5. THE $\tau_2, \widehat{\mathcal{T}}$ RELATION

We obtained (38) by considering an N^L -dimensional vector g which is a direct product of L vectors of dimension N , forming $\tau_2(t_q)g$ and finding the conditions under which this is the sum of two such direct product vectors. There are N^L linearly independent vectors g . The matrix with these columns is $T(x, y)$.

We can also form instead the row-vector $g^T \tau_2(t_q)$. Corresponding working goes through and we are led to define two more Boltzmann weight edge-functions

$$\widehat{W}_J(a|x_J, y_J) = \prod_{i=1}^a \frac{\omega d_{p_{2J}} x_J - \omega^i a_{p_{2J}}}{b_{p_{2J}}/y_J - \omega^{i-1} c_{p_{2J}}}, \quad (39)$$

$$\widehat{\overline{W}}_J(a|x_J, y_J) = \prod_{i=1}^a \frac{\omega a_{p_{2J-1}}/y_J - \omega^{i-1} d_{p_{2J-1}}}{\omega c_{p_{2J-1}} x_J - \omega^i b_{p_{2J-1}}} \quad (40)$$

and a transfer matrix $\widehat{T}(x, y)$ with entries

$$\widehat{T}(x, y)_{\beta a} = \prod_{J=1}^L \widehat{W}_J(\beta_J - a_J | x_J, y_J) \widehat{W}_{J-1}(\beta_{J-1} - a_J | x_{J-1}, y_{J-1}). \quad (41)$$

Setting

$$\widehat{d}(y, t_q) = \prod_{J=1}^L \frac{\omega(y_J - y_{p_{2J}})(t_{p_{2J-1}} - t_q)}{y_{p_{2J-1}} y_{p_{2J}} (y_J - \omega x_{p_{2J-1}})}, \quad (42)$$

we obtain the analogue of (4.21):

$$\widehat{T}(x, \omega y) \tau_2(t_q) = \frac{z(\omega t_q)}{\widehat{d}(\omega y, \omega t_q)} \widehat{T}(x, \omega^2 y) X + \widehat{d}(y, t_q) \widehat{T}(x, y). \quad (43)$$

6. THE τ_j RELATIONS

Here we extend the τ_2 matrices to the set $\tau_1, \dots, \tau_{N+1}$, where $\tau_j = \tau_j(t_q)$ is a polynomial in t_q of degree $(j-1)L$. We shall not give explicit definitions in terms of lattice models, as is done in section 3 of ref. 2, but will use only the relation (38).

We start by defining

$$\begin{aligned} \Delta_r(x, y) &= d(y, t_q) d(\omega y, \omega t_q) \cdots d(\omega^{r-1} y, \omega^{r-1} t_q), \quad r \geq 0 \\ &= \{d(\omega^{-1} y, \omega^{-1} t_q) d(\omega^{-2} y, \omega^{-2} t_q) \cdots d(\omega^r y, \omega^r t_q)\}^{-1}, \quad r \leq 0, \end{aligned}$$

and

$$D_r(x, y) = \Delta_r(x, y) \widehat{T}(\omega x, \omega^r y) T(\omega x, y)^{-1}. \quad (44)$$

In particular, $D_0(x, y) = I$ is the identity matrix. From (29), we expect the RHS to exist and be unique.

The relation (38) can then be written

$$\tau_2(t_q) = z(t_q)X D_{-1}(x, y) + D_1(x, y). \quad (45)$$

The spin-shift operator X commutes with $\tau_2(t_q)$ and with all the $D_r(x, y)$. It is consistent with (45) that all the matrices $D_r(\omega^i x, \omega^j y)$ commute with $\tau_2(t_q)$ and with one another, for all r, i, j . This is what we observe numerically: we shall assume that this is so.

We now define $\tau_j(t_q)$ to be

$$\tau_j(t_q) = \sum_{k=0}^{j-1} z(t_q)z(\omega t_q) \cdots z(\omega^{k-1} t_q) X^k D_k(x, \omega^{k-1} y) D_{j-k-1}(x, \omega^k y) \quad (46)$$

for $j \geq 0$. Then

$$\tau_0(t_q) = 0 \quad \tau_1(t_q) = I$$

and $\tau_2(t_q)$ is as given in (45). From this definition it follows that

$$\tau_2(\omega^{j-1} t_q) \tau_j(t_q) = z(\omega^{j-1} t_q) X \tau_{j-1}(t_q) + \tau_{j+1}(t_q), \quad (47a)$$

$$\tau_j(\omega t_q) \tau_2(t_q) = z(\omega t_q) X \tau_{j-1}(\omega^2 t_q) + \tau_{j+1}(t_q), \quad (47b)$$

$$\tau_{N+1}(t_q) = z(t_q) X \tau_{N-1}(\omega t_q) + \alpha_q + \bar{\alpha}_q, \quad (47c)$$

for $j = 1, \dots, N$, where

$$\alpha_q = \prod_{i=0}^{N-1} d(\omega^i y, \omega^i t_q), \quad (48)$$

$$\bar{\alpha}_q = \prod_{i=0}^{N-1} z(\omega^i t_q) / d(\omega^i y, \omega^i t_q). \quad (49)$$

The two sets of relations (47a), (47b) are equivalent. Note that $\alpha_q, \bar{\alpha}_q$ are unchanged by the mapping

$$x, y, t_q \rightarrow x, \omega y, \omega t_q.$$

These equations are the generalizations of (4.27)–(4.29). In deriving them we have kept $x = \{x_1, \dots, x_L\}$ fixed and incorporated all multiplications by powers of ω into $y = \{y_1, \dots, y_L\}$ and therefore t_q , so if we

write the RHS of (46) more explicitly as $\tau_j(x, y)$, then by $\tau_j(\omega^k t_q)$ in the above equations we mean $\tau_j(x, \omega^k y)$. However, (47a) or (47b) can be used to successively form $\tau_3(t_q)$, $\tau_4(t_q)$, etc. Since $z(t_q)$ and $\tau_2(t_q)$ are polynomials in t_q of degree $2L$ and L , respectively, from this construction it follows that each $\tau_j(t_q)$ is also a polynomial in t_q , of degree $(j-1)L$. So $\tau_j(t_q)$ is indeed a single-valued function of t_q , unchanged by replacing x, y by $\omega x, \omega^{-1}y$, and by the choice of the solution of (23).

From (18) it follows that all the matrices $\tau_j(t_q)$ commute, for all values of t_q . There is therefore a similarity transformation, independent of t_q , that simultaneously diagonalizes all the $\tau_j(t_q)$. Then (47a)–(47c) become scalar functional relations for each eigenvalue, which is also a polynomial in t_q . These relations define the eigenvalue. There are many solutions, corresponding to the different eigenvalues.

Another way of looking at this is to note that if we replace y in (38) by $\omega y, \omega^2 y, \dots, \omega^{N-1}y$, we obtain a total of N homogeneous linear equations for N unknowns $T(\omega x, y), \dots, T(\omega x, \omega^{N-1}y)$. The determinant of these relations must vanish, and that is the relation for the function $\tau_2(t_q)$ obtained by eliminating $\tau_3(t_q), \dots, \tau_{N+1}(t_q)$ from (47a) and (47c), or equivalently from (47b) and (47c).

We have derived the hierarchy of relations (47a)–(47c) from (38). We could equally well have derived them from (43).

7. CALCULATION OF $\alpha_q + \bar{\alpha}_q$

Since each matrix function $\tau_j(t_q)$ is a polynomial in t_q , from (47c), the same must be true of $\alpha_q + \bar{\alpha}_q$, and it must be of degree at most NL . This is by no means obvious: it appears from (48) and (49) that α_q and $\bar{\alpha}_q$ are each quite complicated functions of the solution x_1, \dots, x_L of (23). The object of this section is to unravel this little mystery.

From (16) and (24), using the shorthand notation (22), we can write the condition (23) as

$$\mu_p^N \mu_{p'}^N \frac{(x_{p'}^N - x_{J-1}^N)(t_q^N - x_p^N x_J^N)}{(t_q^N - y_{p'}^N x_{J-1}^N)(y_p^N - x_J^N)} = 1. \quad (50)$$

We can use this relation to eliminate the factors containing y_J in (31) and (48). Using also (36) and (49), it follows that

$$\alpha_q = \prod_{J=1}^L \frac{\mu_p^N \mu_{p'}^N (x_{p'}^N - x_{J-1}^N) (t_p^N - t_q^N)}{y_p^N y_{p'}^N (y_p^N - x_J^N)}$$

$$\bar{\alpha}_q = \prod_{J=1}^L \frac{(y_p^N - x_J^N)(t_{p'}^N - t_q^N)}{y_p^N y_{p'}^N (x_{p'}^N - x_{J-1}^N)}, \quad (51)$$

where in the multiplicands we again write p, p' for p_{2J-1}, p_{2J-2} , respectively.

Since $J = 1, \dots, L$ and $x_0 = x_L$, (50) is a set of L equations for x_1^N, \dots, x_L^N . We noted above that it has two solutions. Let the other solution be $x_1'^N, \dots, x_L'^N$. Then from (50)

$$\begin{aligned} & (x_{p'}^N - x_{J-1}^N)(t_q^N - x_p^N x_J^N)(t_q^N - y_{p'}^N x_{J-1}^N)(y_p^N - x_J'^N) \\ &= (x_{p'}^N - x_{J-1}'^N)(t_q^N - x_p^N x_J'^N)(t_q^N - y_{p'}^N x_{J-1}'^N)(y_p^N - x_J^N). \end{aligned} \quad (52)$$

This equation can be re-written in the ‘‘Wronskian’’ form:

$$\begin{aligned} & (x_{J-1}^N - x_{J-1}'^N)(t_q^N - x_p^N x_J^N)(y_p^N - x_J'^N)(t_{p'}^N - t_q^N) \\ &= (x_J^N - x_J'^N)(t_q^N - y_{p'}^N x_{J-1}^N)(x_{p'}^N - x_{J-1}'^N)(t_{p'}^N - t_q^N). \end{aligned} \quad (53)$$

We can use (50) to eliminate the factors $(t_q^N - x_p^N x_J^N)$, $(t_q^N - y_{p'}^N x_{J-1}^N)$, leaving

$$\frac{(y_p^N - x_J^N)(t_{p'}^N - t_q^N)}{y_p^N y_{p'}^N (x_{p'}^N - x_{J-1}^N)} = \frac{\mu_p^N \mu_{p'}^N (x_J^N - x_J'^N)(x_{p'}^N - x_{J-1}'^N)(t_{p'}^N - t_q^N)}{y_p^N y_{p'}^N (x_{J-1}^N - x_{J-1}'^N)(y_p^N - x_J'^N)}. \quad (54)$$

Taking the product over $J = 1, \dots, L$, the factors $(x_J^N - x_J'^N)$, $(x_{J-1}^N - x_{J-1}'^N)$ cancel, giving

$$\bar{\alpha}_q = [\alpha_q]', \quad (55)$$

where $[\alpha_q]'$ is defined by (51), but with each x_J replaced by x_J' . Interchanging each x_J, x_J' , it follows at once that $\alpha_q = [\bar{\alpha}_q]'$, so $\alpha_q + \bar{\alpha}_q$ is *unchanged* by replacing the solution x_1, \dots, x_L by the alternative solution x_1', \dots, x_L' . It is therefore a single-valued function of t_q .

Now we look at (50), considered as a recursion relation giving x_{J-1}^N in terms of x_J^N . Set

$$x_J^N = f_J / g_J \quad (56)$$

for $J=1, \dots, L$. Then we can choose the normalization so that

$$\begin{aligned} -y_p^N y_{p'}^N f_{J-1} &= (t_q^N - \mu_p^N \mu_{p'}^N x_p^N x_{p'}^N) f_J + (\mu_p^N \mu_{p'}^N x_p^N - y_{p'}^N) t_q^N g_J, \\ -y_p^N y_{p'}^N g_{J-1} &= (y_{p'}^N - \mu_p^N \mu_{p'}^N x_p^N) f_J + (\mu_p^N \mu_{p'}^N t_q^N - y_p^N y_{p'}^N) g_J, \end{aligned} \quad (57)$$

where again $p = p_{2J-1}$, $p' = p_{2J-2}$. This is a linear relation for (f_{J-1}, g_{J-1}) in terms of (f_J, g_J) .

With these definitions, we find that the multiplicand in the second Eq. (51) is simply g_{J-1}/g_J , so

$$\bar{\alpha}_q = \prod_{J=1}^L g_{J-1}/g_J = g_0/g_L. \quad (58)$$

Define two-by-two matrices

$$\begin{aligned} A_{2J} &= \begin{pmatrix} t_q^N & \mu_{p_{2J}}^N x_{p_{2J}}^N \\ y_{p_{2J}}^N & \mu_{p_{2J}}^N \end{pmatrix} / y_{p_{2J}}^N, \\ B_{2J-1} &= \begin{pmatrix} -1 & y_{p_{2J-1}}^N \\ \mu_{p_{2J-1}}^N x_{p_{2J-1}}^N & -\mu_{p_{2J-1}}^N t_q^N \end{pmatrix} / y_{p_{2J-1}}^N \end{aligned} \quad (59)$$

and set

$$\xi_J = \begin{pmatrix} f_J \\ g_J \end{pmatrix}.$$

Then (57) can be written as

$$\xi_{J-1} = A_{2J-2} B_{2J-1} \xi_J. \quad (60)$$

Since $x_0^N = x_L^N$, it follows that $\xi_0 = \lambda \xi_L$, where

$$\xi_0 = \lambda \xi_L = A_{2L} B_1 A_2 B_3 \cdots A_{2L-2} B_{2L-1} \xi_L \quad (61)$$

Thus λ is the eigenvalue of $U = A_{2L} B_1 \cdots A_{2L-2} B_{2L-1}$, ξ_L is the corresponding eigenvector and, from (58),

$$\bar{\alpha}_q = \lambda. \quad (62)$$

Since U is a two-by-two matrix, it has two eigenvalues λ and λ' , corresponding to the two solutions x and x' of the recurrence relations. However, we have just shown that interchanging the solutions replaces $\bar{\alpha}_q$ by α_q , so

$$\alpha_q = \lambda'. \quad (63)$$

Since $\lambda + \lambda'$ is the trace of the matrix U , it follows that

$$\alpha_q + \bar{\alpha}_q = \text{Trace} (A_{2L} B_1 A_2 B_3 \cdots A_{2L-2} B_{2L-1}). \quad (64)$$

We are regarding the vertical rapidity parameters $x_{p_1}, y_{p_1}, \mu_{p_1}, \dots, x_{p_L}, y_{p_L}, \mu_{p_L}$ as constants and t_q as a complex variable, so this is an explicit expression for $\alpha_q + \bar{\alpha}_q$ that makes it clear that it is indeed a polynomial in t_q^N . Since $A_{2J-2} B_{2J-1}$ is linear in t_q^N , this polynomial is of degree not greater than L (in general it is of degree L).

From (36), (51) and (59) it is readily seen that

$$\lambda \lambda' = \det U = \prod_{i=1}^N z(\omega^i t_q) = \alpha_q \bar{\alpha}_q,$$

so we could have obtained (63) without going through the working from Eqs. (52) to (55). We have included that working, partly for completeness, but also because it is an elegant example of how in solvable models the algebra conspires to produce needed results.

The $\tau_j(t_q)$ relations (47a)–(47c), together with (59) and (64), provide a closed set of equations that determine the eigenvalues of the $\tau_j(t_q)$ matrices, all quantities being polynomials in the complex variable t_q . To use them, there is no need to solve the eigenvalue equation (61), which is equivalent to the recurrence relation (23). We could presumably have obtained these relations directly by a “fusion” method, generalizing the definition (3.26)–(3.44) of $\tau_j(t_q)$, but this is quite technical. We prefer the present approach, based on the Eq. (38).

We have assumed that the matrices $D_r(\omega^i x, \omega^j y)$ commute with one another and with $\tau_2(t_q)$. This assumption agrees with numerical calculations we have performed for $N = L = 3$, but it can probably be removed. We can certainly apply a similarity transformation (independent of t_q) that diagonalizes $\tau_2(t_q)$ (for all t_q). Applying this only to the left of (38), it becomes a set of many equations for each eigenvalue of $\tau_2(t_q)$. If we focus on just one eigenvalue and one such equation, then, as we remarked above, we can obtain N relations from it by replacing y, t_q by $\omega^i, \omega^i t_q$, for $i = 0, \dots, N - 1$. These are homogeneous linear relations for the corresponding N elements of $T(\omega x, \omega^j y)$, so their determinant must vanish. The resulting determinantal relation must be equivalent to (47a)–(47c).

8. THE T, \widehat{T} RELATIONS

From the definitions (46) we can also establish that

$$\alpha_q \tau_j(t_q) + z(t_q) \cdots z(\omega^{j-1} t_q) X^j \tau_{N-j}(\omega^j t_q) = D_j(x, \omega^{-1} y) \tau_N(t_q)$$

for $j=0, \dots, N$. From (44), there must therefore be a matrix $Y(x, y)$, independent of j , such that

$$\begin{aligned} \Delta_j(x, \omega^{-1} y) T(\omega x, \omega^j y) Y(x, y) \\ = \alpha_q \tau_j(t_q) + z(t_q) \cdots z(\omega^{j-1} t_q) X^j \tau_{N-j}(\omega^j t_q) \end{aligned} \quad (65)$$

for $j=0, \dots, N$.

These equations have the same structure as the fusion hierarchy of relations (3.46), except we still have to identify the matrix Y . We have not fully done this, but we can note from (25), (26), (39), (40) that

$$\overline{W}_J(a|\omega x_J, y_J) \widehat{W}_J(-a|x_J, y_J) = 1, \quad (66)$$

$$\begin{aligned} \sum_c W_J(a-c|\omega x_J, y_J) \widehat{W}_J(c-b|x_J, y_J) \\ = \frac{N(x_p - \omega^{-1} y_J)(y_p - \omega x_J)(t_p^N - t_q^N)}{(t_p - t_q)(x_p^N - y_J^N)(y_p^N - x_J^N)} \delta_{a,b}, \end{aligned} \quad (67)$$

where $p = p_{2J-1}$, and $\delta_{a,b} = 1$ if $a = b$ to modulo N , else $\delta_{a,b} = 0$. The sum is over any N consecutive integer values of c : although the W functions individually are not periodic functions, the products in the above two equations are indeed periodic functions of a, b, c , of period N .

Consider the matrix product $T(\omega x, y) \widehat{T}(x, y)$, by which we mean the usual sum over the intermediate indices, in this case both being the β indices in (28), (41). Incrementing any β_J by N multiplies the columns of $T(\omega x, y)$ by certain factors, but divides the rows of $\widehat{T}(x, y)$ by the same factors, so leaves their product unchanged. Thus we can naturally take the intermediate sum to be over the values $0, \dots, N-1$ (or any set of N successive values) of each of β_1, \dots, β_L .

The matrix product is the Boltzmann weight of two successive rows of the lattice. A typical face of this double row is shown in Fig. 6. Suppose the two external spins a_{j+1}, b_{j+1} are equal. Then from (66), the Boltzmann weight factors $\overline{W}_J \widehat{W}_J$ factors cancel. The sum over the centre spin c then gives the RHS of (67), which vanishes unless $a_J = b_J, \text{ mod } N$.

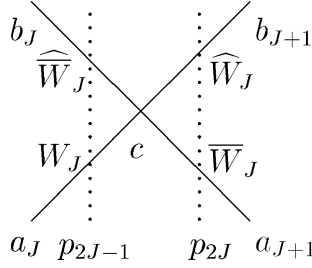


Fig. 6. The Boltzmann weight of a face of the transfer matrix product $T(\omega x, y)\widehat{T}(x, y)$.

If $a_J = b_J$, then the same argument applied to the face to the left tells us that $a_{J-1} = b_{J-1}$, and so on. It follows that

$$T(\omega x, y)\widehat{T}(x, y) = [\tau_1(t_q) + R] / g(x, y), \tag{68}$$

where R is a matrix with non-zero elements R_{ab} only when $a_1 \neq b_1, \dots, a_L \neq b_L$, and

$$g(x, y) = \prod_{J=1}^L \frac{(t_p - t_q)(x_p^N - y_J^N)(y_p^N - x_J^N)}{N(x_p - \omega^{-1}y_J)(y_p - \omega x_J)(t_p^N - t_q^N)}, \tag{69}$$

where each p in the multiplicand is p_{2J-1} .

This is an ‘‘inversion identity’’: it has the same structure as the $j = 1$ case of (65), with the matrix $T(\omega x, y)$ on the left and the first term on the right being proportional to $\tau_1(t_q)$, i.e. to the identity matrix. We conjecture (in agreement with numerical calculations) that the RHS of the two relations are in fact the same, to within a scalar factor, in which case $R = z(t_q)X\tau_{N-1}(\omega t_q)/\alpha_q$ and

$$\Delta_1(x, \omega^{-1}y)Y(x, y) = \alpha_q g(x, y)\widehat{T}(x, y). \tag{70}$$

Then (65) becomes

$$\begin{aligned} g(x, y) \Delta_{j-1}(x, y)T(\omega x, \omega^{j-1}y)\widehat{T}(x, y) \\ = \tau_j(t_q) + z(t_q) \cdots z(\omega^{j-1}t_q)X^j\tau_{N-j}(\omega^j t_q)/\alpha_q. \end{aligned} \tag{71}$$

This is the generalization of (3.46).

8.1. Consistency

An interesting consistency check on (71) is provided by post-multiplying (38) by $\widehat{T}(x, \omega y)$, and pre-multiplying (43) by $T(\omega x, y)$. The LHS are then the same, equating the RHS gives

$$\begin{aligned} & \frac{z(t_q)}{d(\omega^{-1}y, \omega^{-1}t_q)} X T(\omega x, \omega^{-1}y) \widehat{T}(x, \omega y) + d(y, t_q) T(\omega x, \omega y) \widehat{T}(x, \omega y) \\ &= \frac{z(\omega t_q)}{\widehat{d}(\omega y, \omega t_q)} T(\omega x, y) \widehat{T}(x, \omega^2 y) X + \widehat{d}(y, t_q) T(\omega x, y) \widehat{T}(x, y). \end{aligned} \quad (72)$$

We can use (71) to express each of the $T\widehat{T}$ products as a sum of τ_j terms. In fact we get only terms proportional to $\tau_1(t_q)$, $\tau_1(\omega t_q)$, $\tau_{N-1}(\omega t_q)$ and $\tau_{N-1}(\omega^2 t_q)$. There are two terms proportional to each of these four factors. Since $\tau_1(t_q) = I$, we can interchange $\tau_1(t_q)$ with $\tau_1(\omega t_q)$ on the RHS. Using only the relations

$$\begin{aligned} \Delta_{N-2}(x, \omega y) &= \alpha_q / \{d(y, t_q) d(\omega^{-1}y, \omega^{-1}t_q)\}, \\ \widehat{d}(y, t_q) &= d(y, t_q) g(x, y) / g(x, \omega y) \end{aligned}$$

and the commutation of X with all the τ_j matrices, we then find that the two terms for each τ factor cancel, thereby verifying (72).

9. THE $\tau_2(t_q)$ MODEL WITH OPEN BOUNDARIES

We return to considering the hierarchy of relations (47a)–(47c) for the $\tau_j(t_q)$ functions.

These relations simplify greatly when we impose fixed-spin boundary conditions on the left and right sides of the lattice. We can do this by taking

$$a_{p_1} = d_{p_1} = 0. \quad (73)$$

Then $F_{p_1 q}(1, m) = 0$, so the weight function of Fig. 4 vanishes for the faces between column 1 and column 2 unless $a = d$. This is equivalent to requiring that all the spins in column 1 of Fig. 1 be equal. The model is unchanged by incrementing every spin by one, so we can in particular require that every spin on column 1 be zero. It is evident from Fig. 1 that this is the same as requiring that all spins on the left and right boundaries be zero.

Equation (73) implies that

$$\mu_{p_1} = x_{p_1} = z(t_q) = \bar{\alpha}_q = 0 \quad (74)$$

so the relations (47a), (47b) imply that

$$\tau_j(t_q) = \tau_2(t_q)\tau_2(\omega t_q) \cdots \tau_2(\omega^{j-2}t_q), \quad (75)$$

for $j = 1, \dots, N + 1$. Hence (47c) simplifies to

$$\tau_2(t_q)\tau_2(\omega t_q) \cdots \tau_2(\omega^{N-1}t_q) = \alpha. \quad (76)$$

From (64), noting that the second row of the matrix B_1 is now zero,

$$\alpha = [B_1 A_2 B_3 A_4 \cdots B_{2L-1} A_{2L}]_{11}. \quad (77)$$

The RHS is a polynomial in t_q^N of degree L and we noted above that $\tau_2(t_q)$ and its eigenvalues are polynomials in t_q of degree L . Further, when either t_q is large or small, to leading order $\tau_2(t_q)$ is diagonal, with entries

$$\prod_{J=1}^L (1 - \omega^{a_{J+1} - a_J + 1} t_q / y_{2J-1} y_{2J})$$

in row and column $a = \{a_1, \dots, a_L\}$. Let the zeros of (77) be $s_1^N, s_2^N, \dots, s_L^N$. Then it follows that all eigenvalues of $\tau_2(t_q)$ are of the form

$$\Lambda(t_q) = (\omega^L / Y) \prod_{j=1}^L (s_j - \omega^{\gamma_j} t_q), \quad (78)$$

where $s_1 s_2 \cdots s_L = \omega^{-L} Y$, and

$$Y = \prod_{J=1}^L y_{p_{2J-1}} y_{p_{2J}}. \quad (79)$$

The $\gamma_1, \dots, \gamma_L$ are integers with values in the range $0, \dots, N - 1$. They satisfy the condition $\gamma_1 + \dots + \gamma_L = 0$, and it seems from low-temperature expansions that the full set of N^{L-1} eigenvalues is obtained by allowing $\gamma_1, \dots, \gamma_L$ to take all such values (distinct to modulo N).

It appears that the other values of $\gamma_1, \dots, \gamma_L$ that do not satisfy the sum rule also correspond to eigenvalues of $\tau_2(t_q)$, provided we generalize the model to allow the skewed boundary conditions

$$a_{L+1} = a_1 + r$$

in every row of the lattice (so all spins in column 1 are still the same, as are all spins in column $L + 1$, but now those in the two boundary columns no longer need be equal). Then

$$\gamma_1 + \dots + \gamma_L = r.$$

We can take r to be an integer in the range $0, \dots, N - 1$.

The eigenvalues therefore have the same simple structure as do direct products of L matrices, each of size N by N . For $N = 2$ this is the structure of the eigenvalues of the Ising model.⁽⁹⁾

For the Ising model this property follows from Kaufman's solution in terms of spinor operators,⁽¹⁰⁾ i.e. a Clifford algebra.^(11, p. 189) Whether there is some generalization of such spinor operators to handle the $\tau_2(t_q)$ model with open boundaries remains a fascinating speculation.⁽¹²⁾

The results of this section were anticipated in ref. 5. There we considered the superintegrable chiral Potts model and rotated it though 90° to obtain a model that is in fact the present $\tau_N(t_q)$ model. Then we inverted its row-to-row transfer matrix, thereby obtaining the present $\tau_2(t_q)$ model.⁶ We did in fact note in section 7 of ref. 5. that we could allow the modulus k to be different for different rows: this corresponds to our here allowing a_p, b_p, c_p, d_p to all vary arbitrarily from column to column.

10. SUMMARY

We have shown that the column-inhomogeneous $\tau_2(t_q)$ model is solvable for *all* values of the $8L$ parameters $a_{p_1}, b_{p_1}, c_{p_1}, d_{p_1}, \dots, d_{p_{2L}}$, where $a_{p_J}, b_{p_J}, c_{p_J}, d_{p_J}$ are associated with the J th vertical dotted line in Fig. 1. They do *not* need to satisfy the "chiral Potts" conditions (1). The model then has the unusual property that its row-to-row transfer matrices (with different values of t_q but the same $a_{p_1}, \dots, d_{p_{2L}}$) commute, while the column-to-column transfer matrices do not.

Our results (38), (43), (47a)–(47c), (71) generalize the relations (4.20), (4.21), (4.27a)–(4.27c), (3.46) of ref. 2. The last generalization (71) is essentially a conjecture, depending as it does on the identification of (68) with

⁶Note from (75) that $\tau_2(t_q)\tau_N(\omega t_q) = \alpha$, so to within a scalar factor the transfer matrix $\tau_2(t_q)$ is the inverse of $\tau_N(\omega t_q)$.

the $j=1$ case of (65). However, it has been tested numerically for $N=L=3$ with arbitrarily chosen values of the parameters and found to be true to the 30 digits of precision used.

One significant difference from the homogeneous model is that the associated chiral Potts model weights (25), (26), (39), (40) depend on t_q via the solution r_1, \dots, r_L of (23). If we change t_q then we change r_1, \dots, r_L in a non-trivial way, so it makes little sense to combine $T(x, y)$ for one value of t_q with $\hat{T}(x, y)$ for another value. It appears that our generalized chiral Potts transfer matrices T and \hat{T} , with different values of t_q , do not satisfy any general commutation relations like (2.31)–(2.33) of ref. 2.

In short, we can generalize the $\tau_2(t_q)$ model to arbitrary $a_{p_1}, \dots, d_{p_{2L}}$, but the only chiral Potts model we can correspondingly generalize is the “superintegrable” model with the alternate row-to-row transfer matrices $T(x, y)$, $\hat{T}(x, y)$ defined above. In each double row t_q, r_1^N, \dots, r_L^N must be the same for $T(x, y)$ and $\hat{T}(x, y)$.

The functional relations (47a)–(47c) define the eigenvalues of the row-to-row transfer matrix $\tau_2(t_q)$. For fixed-spin conditions on the left and right boundaries these can be solved explicitly, giving the simple “direct product” result (78).

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